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COLLAP SE, BUCKLING AND POST FAILURE

OF CYLINDRICAL SHELLS

Joshua E. Greenspon, Dr. Eng.

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ABSTRACT

In this report it is demonstrated how simple existing theories can predict whether a shell will fail by buckling or form a collapse hinge and then fail by this collapse mechanism. It is also shown how work-energy analysis can predict the post failure plastic deflections of the shell once the mode of failure is known. Some comparisons with experimental results are given.

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LIST OF SYMBOLS

$N_\phi, N_{x\phi}, N_x$ stress resultants

x longitudinal coordinate

ϕ peripheral coordinate

$f_1(\phi), f_2(\phi),$ arbitrary functions of ϕ which are determined from the
$f_3(\phi), f_4(\phi)$ boundary conditions

a mean radius of the shell

$p(x, \phi)$ pressure distribution

h thickness of shell

u, v, w longitudinal, tangential and radial displacements respectively

α, β parameters used to define the peripheral pressure amplitude

ν Poisson's ratio

E modulus of elasticity

ℓ length of shell

n number of circumferential waves in buckling pattern

σ_0 yield stress in pure tension

$(p_0)_c$ pressure which initiates collapse

$(q_{cr})_b$ pressure which initiates buckling

ρ mass density of shell material

$g(x, y)$ spatial distribution of the impulse

V work done by shell in deforming

w_0 maximum deflection amplitude

$\mathcal{F}(x', \phi)$ spatial distribution of the deflection

e_0 yield strain in pure tension

x' x/ℓ

d_0 width of plastic hinge

K defined by Eq. [20]

I_t total impulse (theoretical)

$\bar{I}$ defined by Eq. [22]

$\bar{I}_e$ experimental impulse per unit area

$(\bar{I}_e)_r$ reflected experimental impulse per unit area

$(\bar{I}_e)_i$ incident experimental impulse per unit area

k parameter defining periphery die out of spatial deflection distribution in buckling

I. Introduction

A previous report[1]*presented the plastic work expressions for cylindrical shells and discussed the types of failure that could occur in cylindrical shells under blast loading. Experiments have shown[2] that cylindrical shells subjected to side-on blast can go into two main types of failure. These are buckling and collapse. The buckling type of failure is described by a deformation pattern which consists of a number of lobes around the periphery of the shell and one half wave length along the length as shown in Figure la. The collapse failure is described by a straight failure hinge at the center of the shell as shown in Figure lb. Both of these figures are taken from Schuman's experimental results.[2] The type of failure will depend upon the geometry of the shell and can be predicted from an elastic stress and buckling analysis of the shell as will be seen later in this report.

II. Elastic stress analysis

Assume that the shell is thin and that membrante theory is adequate to describe the stress patterns in the shell. Assume also that the shell is of length ℓ and is supported at each end. Take the origin of coordinates at the center of the shell as shown in Figure 2. The membrane forces are shown on the differential element in Figure 3.

If $p(x, \phi)$ is the static load per unit area applied laterally to the shell then it follows from the basic membrane equations[3,4] that

$$
\begin{aligned}
N_\phi &= a\, p(x, \phi) \\
N_{x\phi} &= -\int \frac{1}{a} \frac{\partial N_\phi}{\partial \phi} dx + f_1(\phi) \\
N_x &= -\int \frac{1}{a} \frac{\partial N_{x\phi}}{\partial \phi} dx + f_2(\phi)
\end{aligned}
$$

[1]

where $f_1(\phi)$ and $f_2(\phi)$ are functions of ϕ which are to be determined from the boundary conditions on $N_x, N_{x\phi}, N_\phi$. If some of the boundary conditions are given in terms of displacements then the following membrane equations in terms of displacements must be utilized[3]

---

*Superscripts refer to references given at the back of the report.

$$EhU = \int (N_x - \nu N_\varphi)dx + f_3(\varphi)$$

$$Ehv = 2(1+\nu)\int N_{x\varphi}dx - \frac{Eh}{a}\int \frac{\partial u}{\partial \varphi}dx + f_4(\varphi) \qquad [2]$$

$$Ehw = a(N_\varphi - \nu N_x) + Eh \frac{\partial v}{\partial \varphi}$$

where h is the shell thickness, E is the modulus of elasticity, U, v, w are the displacements (see Fig. 3) and $f_3(\varphi)$, $f_4(\varphi)$ are arbitrary functions to be determined from the boundary conditions.

Now assume that the pressure $p(x,\varphi)$ can be represented as

$$p(x,\varphi) = p_0 e^{-\beta\varphi}\cos\alpha\varphi \qquad 0 < \varphi < \pi$$

$$= p_0 e^{-\beta(2\pi-\varphi)}\cos\alpha(2\pi-\varphi) \quad \pi < \varphi < 2\pi \qquad [3]$$

where p_0 is the maximum amplitude of the pressure and β, α are parameters determining the peripheral pressure distribution. If the shell length is small compared to the distance from the explosion and if the blast is a side-on load then it seems reasonable that the _spatial load distribution_ given by [3] should resemble the spatial distribution produced by the blast.

The solution will be obtained for the boundary conditions

$$U = 0 \quad \text{at} \quad x = \pm \ell/2$$

$$(N_{x\varphi} = 0 \text{ at } x = 0 \text{ by symmetry}) \qquad [4]$$

By straight forward integration of equations [1] and [2] subject to boundary conditions [4] it is found that

$$N_\varphi = a p_0 f(\varphi), \quad f(\varphi) = e^{-\beta\varphi}\cos\alpha\varphi \qquad [5]$$

$$N_{x\varphi} = -p_0 x \qquad f'(\varphi) = \frac{df}{d\varphi}, \quad f''(\varphi) = \frac{d^2 f}{d\varphi^2}$$

$$N_x = \frac{p_0}{a}\frac{x^2}{2}f''(\varphi) + \nu a\, p_0 f(\varphi) - \frac{p_0}{a}\frac{\ell^2}{24}f''(\varphi)$$

The stress resultants at $x = \varphi = 0$ are

$$N_\varphi = a p_0$$

$$N_{x\varphi} = 0 \qquad [6]$$

$$N_x = \nu a p_0 - \frac{p_0}{a}\frac{\ell^2}{24}(\beta^2 - \alpha^2)$$

Yielding will occur at $x = \varphi = 0$ when the Von Mises Yield Conditon is satisfied at this point, i.e.

$$N_x^2 - N_x N_\varphi + N_\varphi^2 + 3 N_{x\varphi}^2 = \sigma_0^2 h^2 \qquad [7]$$

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where σ_o is the yield stress in pure tension. Substituting the values of the stress resultants into the yield condition it is found that yielding will occur at the center of the shell ($x = 0, \phi = 0$) when the pressure p_o has a value

$$(p_o)_c = \sigma_o \frac{h}{a} \frac{1}{\sqrt{(1-\nu+\nu^2) + (\beta^2-\alpha^2)\left[\frac{\ell^2}{24a^2}(1-2\nu) + (\frac{\ell^2}{24a^2})^2(\beta^2-\alpha^2)\right]}} \quad [8]$$

III. Elastic buckling analysis

The classical theory of buckling of cylindrical shells is presented by Timoshenko.[5] It is found that the lateral pressure at which buckling under uniform loading will occur is given by the following relation[6]

$$(q_{cr})_b = \frac{Eh}{a(1-\nu^2)}\left[\frac{1-\nu^2}{(n^2-1)(1+\frac{n^2\ell^2}{\pi^2a^2})^2} + \frac{h^2}{12a^2}\left(n^2-1+\frac{2n^2-1-\nu}{1+\frac{n^2\ell^2}{\pi^2a^2}}\right)\right] \quad [9]$$

where n is the number of full waves around the periphery of the shell. It has been found by Reynolds[7] that in lobar buckling such as this the circumferential parameter n approximately satisfies the relation

$$\frac{1}{1+\frac{n^2\ell^2}{\pi^2a^2}} \approx 1.23 \frac{\sqrt{ah}}{\ell} \quad [10]$$

and that buckling will always occur with only one half wave along the length. In other words, if m denotes the number of axial half waves along the length, then $m = 1$ and n is determined from equation [10]. The factor 1.23 has to be adjusted so that n turns out as a whole number. Once n is determined then equation [9] will give the value of the buckling load for uniform loading. For nonuniform loading correction factors can be applied in accordance with a recent paper.[8]

IV. Buckling or yield

It is clear from Schuman's experiments[2] that both buckling and yield collapse can occur. The main problem is to be able to predict which type will take place. Once buckling or collapse has commenced the plastic deformation will take place in that particular pattern into which the failure has started. It will now be shown how equation [8] and [9] can be employed to determine whether a collapse or buckling will take place. Although there were some gross assumptions involved in the derivation of equation [8] the main item to be recognized is that the pressure for this type of yield is proportional to h/a and dependent on $\nu, \ell/a$. On the other hand the buckling pressure is critically dependent on $n, h/a$ and h^2/a^2 as well as $\nu, \ell/a$.

It should be made clear at this point that equation [8] and [9] are not to be used to approximate any of the dynamical parameters of the shell; they are to be used only to determine which type of failure

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will start. Once this is known then the plastic analysis as given in a previous reference[1] will be employed to determine the plastic deflections and impulse values. Assuming that

$$E \approx 1000 \, \sigma_o \, , \quad \beta = 2, \quad \alpha = \frac{1}{2}$$

it was found that shell buckling or yield could be predicted by the equations in the previous section by employing the following criterion:

> If the buckling load is less than the yield load the shell should buckle; if the yield load is less than the buckling then the shell should collapse.

Due to lack of experimental evidence on the load distribution the parameters β and α are questionable. Perhaps when more information is available about the load distribution more accurate computations can be performed. However our present calculations do indicate that the simplified equations presented above describe the physical phenomenon and enable prediction of the type of failure that will occur in a given cylindrical shell.

To illustrate the procedure for determining whether collapse or buckling failure will occur, consider several cylindrical shells that were tested.

1. Consider a steel shell with radius of 1.5", length 11.62", thickness .019" subjected to a lateral blast which is 8' away. The experiment showed that this shell formed a collapse hinge.

   Using an $E = 1000 \, \sigma_o$, $\beta = 2$, $\alpha = \frac{1}{2}$, $\nu = .3$ the collapse load according to [8] is

   $$(P_o)_c \approx .0013 \, \sigma_o$$

   If the shell would buckle [10] predicts that $n = 3$. Then equation [9] shows that the uniform buckling load for this n is

   $$(q_{cr})_b \approx .0021 \, \sigma_o$$

   and using the nonuniform pressure correction[8] it is seen that this value of .0021 σ_o will be somewhat higher, possible as much as 50% greater. Thus for this shell $(P_o)_c < (q_{cr})_b$ so that

   the shell can be expected to fail by collapse.

2. Consider next an aluminum shell of radius 1.5", length 6", thickness .006" subjected to a lateral blast which is 15' away. The experiment showed that this shell buckled with 6 lobes around the periphery and a half wave along the length. Using an $E = 1000 \, \sigma_o$ $\beta = 2$, $\alpha = \frac{1}{2}$, $\nu = .3$, the yield load would be

   $$(P_o)_c \approx .0014 \, \sigma_o$$

   Under buckling [10] predicts $n = 6$. Equation [9] shows that the uniform buckling pressure will be

   $$(q_{cr})_b \approx .00024 \, \sigma_o$$

   The correction for nonuniform loading could possible raise this value to .00036 σ_o. Thus for this shell $(q_{cr})_b < (P_o)_c$ so

   that the shell can be expected to buckle.

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V. Post failure collapse and buckling

   A. Collapse

It has been found[1] that the relation between the total impulse applied to the shell and the work done on the shell can be written as

$$I_t = \int_A \sqrt{V \frac{2 \rho h}{\int_A g^2(x,y)\, dA}}\; g(x,y)\, dA \qquad [11]$$

where $g(x,y)$ is the spatial distribution of the impulse, ρ is the mass density of the shell material, h is the shell thickness, and V is the work done on the shell in deforming it plastically. The integral is taken over the area, A, of the shell. By a systematic simplification it has been shown[1] that V can be written as

$$V \approx \frac{2 \sigma_o h w_o \ell}{\sqrt{3}} \int_o^1 \int_o^{2\pi} \bar{f}(x', \phi)\, d\phi\, dx' \qquad [12]$$

where σ_o is the yield stress in pure tension, w_o is the maximum deflection, h is the thickness, ℓ is the length and $f(x', \phi)$ is the spatial distribution of the deflection, i.e.

$$w = w_o \bar{f}(x', \phi)$$

The assumptions under which [12] was derived are

1. The shell is made of a perfectly plastic material
2. The radius of the shell is considerably smaller than the length ($a/\ell << 1$)
3. The deflection of the shell is much smaller than the radius ($w_o/a << 1$)
4. $w_o/\sqrt{3} >> e_o a / 2$ where e_o is the yield strain

For details see Reference 1

The deflection pattern for collapse is shown in Figure 4 and can be written analytically as follows:

$$w(x, \phi) = a \cos\phi - \sqrt{a^2 - \tfrac{1}{4}\left[d_o - \frac{d_o}{\ell_{/2}} x \right]^2} \qquad [13]$$

Letting $x' = x/\ell$

$$w(x, \phi) = a \cos\phi - \sqrt{a^2 - \frac{d_o^2}{4}[1 - 2x']^2} \qquad \text{for } 0 < x' < \tfrac{1}{2}$$
$$= a \cos\phi - \sqrt{a^2 - \frac{d_o^2}{4}[1 + 2x']^2} \qquad \text{for } -\tfrac{1}{2} < x' < 0 \qquad [14]$$

where d_o is the width of the hinge line as shown in Figure 4. Using these deformation expressions and [12], the work done on the shell in deforming it plastically can be written as

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$$V = \frac{2\sigma_0 h \ell}{\sqrt{3}} \left\{ \int_{-\frac{1}{2}}^{0} \int_{-\phi_1(x')}^{\phi_1(x')} \left[a\cos\phi - \sqrt{a^2 - \frac{d_0^2}{4}[1+2x']^2} \right] d\phi\, dx' \right.$$

$$\left. + \int_{0}^{\frac{1}{2}} \int_{-\phi_1(x')}^{\phi_1(x')} \left[a\cos\phi - \sqrt{a^2 - \frac{d_0^2}{4}[1-2x']^2} \right] d\phi\, dx' \right\} \qquad [15]$$

where

$$\phi_1 = \sin^{-1}\frac{d_0(1-2x')}{2a} \qquad \text{for } 0 < x' < \tfrac{1}{2}$$

$$\phi = \sin^{-1}\frac{d_0(1+2x')}{2a} \qquad \text{for } -\tfrac{1}{2} < x < 0 \qquad [16]$$

After some mathematical manipulation and substitution of

$$\mathscr{y} = \frac{d_0}{2a}[1+2x'] \quad , \quad \mathscr{y}' = \frac{d_0}{2a}[1-2x'] \qquad [17]$$

it is found that V can be written

$$V = \frac{2\sigma_0 h \ell}{\sqrt{3}} \frac{4a}{2\frac{d_0}{2a}} \int_{0}^{d_0/2a} \left(\mathscr{y} - \sqrt{1-\mathscr{y}^2}\,\sin^{-1}\mathscr{y} \right) d\mathscr{y} \qquad [18]$$

or finally

$$V = \frac{2\sigma_0 h \ell d}{\sqrt{3}} \frac{1}{\frac{d_0}{d}} \left[\frac{3}{4}\left(\frac{d_0}{d}\right)^2 - \frac{\bar{\phi}^2}{4} - \frac{\bar{\phi}}{2}\frac{d_0}{d}\sqrt{1-\left(\frac{d_0}{d}\right)^2} \right] \qquad [19]$$

where $\quad \bar{\phi} = \sin^{-1}\frac{d_0}{d} \qquad d = 2a$

Now using the impulse equation [11] and assuming that

$$g(x,\mathscr{y}) \equiv g(x,\phi) = e^{-\beta\phi}$$

The relation between the impulse and the deformation can be written as

$$I_t = 2h\ell d \sqrt{K} \sqrt{\frac{\sigma_0 \rho}{\sqrt{3}}} \qquad [20]$$

where $\quad K = \frac{1}{\frac{d_0}{d}} \left[\frac{3}{4}\left(\frac{d_0}{d}\right)^2 - \frac{\bar{\phi}^2}{4} - \frac{\bar{\phi}}{2}\frac{d_0}{d}\sqrt{1-\left(\frac{d_0}{d}\right)^2} \right]$

The deflection is actually described by d_0/d. A plot of $\sqrt{K}$ as a function of d_0/d is given in Fig. 5.

Some calculations were run on several of the tested shells which failed by collapse and these computations with the experimental results are given below. A given value of deflection corresponding to a given d_0/d was measured from the tests and the incident and reflected impulse per unit area corresponding to this d_0/d were also measured. The theory (eq. [20] above) predicts the impulse which will produce a given d_0/d. Unfortunately the exact value of the experimental impulse was not given but the incident impulse*per unit

*The incident impulse is the measured impulse without an obstruction such as the shell. The reflected impulse is the impulse that reflects off of a large rigid plane.

area $(\bar{I}_e)_i$; and the reflected impulse per unit area $(\bar{I}_e)_r$ were the parameters that were measured. The actual impulse applied to the shell will be somewhere between these values of $(\bar{I}_e)_i$ and $(\bar{I}_e)_r$.

Assuming that the measured values would correspond to maximum values on the shell and assuming an $e^{-\beta\phi}$ distribution of the impulse on the shell it is seen that the total impulse would be

$$(I_t)_e = 2 \int_0^\ell \int_0^\pi (\bar{I}_e) \, e^{-\beta\phi} a \, d\phi \, dx \qquad [21]$$

So $\qquad (\bar{I})_e \approx 2 \dfrac{(I_t)_e}{d\ell}$

So the theoretical value to compare with the measurements is

$$\bar{I} = 2 \frac{I_t}{d\ell} = 4h \sqrt{k} \sqrt{\frac{\sigma_0 \rho}{\sqrt{3}}} \qquad [22]$$

A value of $\sigma_0 \approx 50,000 \, psi$ was assumed for the steel shells that were tested

Example 1

$$d = 3'', \quad \ell = 6'', \quad h = .019''$$
$$\text{For } d_0/d \approx 1 \quad \sqrt{k} \approx .36 \text{ from Fig. 5}$$
$$\text{So } \bar{I} \approx 123 \, psi \, millisec.$$

The measured values which gave the shell a deformation in which $d_0/d \approx 1$ were $\qquad (\bar{I}_e)_i = 89 \, psi \, millisec., \quad (\bar{I}_e)_r = 299 \, psi \, millisec.$

Example 2

$$d = 3'', \quad \ell = 8.62'', \quad h = .019''$$
$$\text{For } d_0/d \approx .8, \quad \sqrt{k} \approx .24$$
$$\bar{I} \approx 82 \, psi \, millisec.$$

The measured values were

$$(\bar{I}_e)_i = 15.5 \, psi \, millisec., \quad (\bar{I}_e)_r = 69.5$$

Example 3

$$d = 3'', \quad \ell = 8.62'', \quad h = .019''$$
$$\text{For } d_0/d \approx .75, \quad \sqrt{k} \approx .21$$
$$\bar{I} = 71.5 \, psi \, millisec.$$

The measured values were

$$(\bar{I}_e)_i = 48, \quad (\bar{I}_e)_r = 160$$

Example 4
$$d = 3'', \quad \ell = 11.62'', \quad h = .019''$$
$$\text{For } d_0/d \approx .75 \quad \sqrt{k} \approx .21 \quad \bar{I} = 71.5 \, psi \, millisec.$$
The measured values were
$$(\bar{I}_e)_i = 14.5, \quad (\bar{I}_e)_r = 52.1$$

Example 5 $d = 3''$, $\ell = 11.62''$, $h = .019''$

For $d_0/d \approx .75$, $\sqrt{K} \approx .21$, $\bar{I} \approx 71.5$

The measured values were

$$(\bar{I}_e)_i = 25 \quad (\bar{I}_e)_r = 83$$

Example 6 $d = 3''$, $\ell = 18''$, $h = .019''$

For $d_0/d \approx 1$, $\bar{I} \approx 123$

The measured values were

$$(\bar{I}_e)_i = 68 \quad (\bar{I}_e)_r = 190$$

Example 7 $d = 6''$, $\ell = 18''$, $h = .019''$

For $d_0/d \approx .75$, $\bar{I} \approx 71.5$

The measured values were

$$(\bar{I}_e)_i = 22 \quad (\bar{I}_e)_r = 67$$

Example 8 $d = 6''$, $\ell = 17.5''$, $h = .035''$

For $d_0/d \approx .75$, $\bar{I} \approx 143$

The measured values were

$$(\bar{I}_e)_i = 64 \quad (\bar{I}_e)_r = 200$$

The different impulse values arose from various explosive charge weights and distances of explosion from the shell.

The comparisons between the theoretical predictions and experimental results show that the simplified equations predict the correct order of magnitude for the impulse which will result in a given deflection. When more exact experimental values are available for the applied impulse more accurate computations using the more accurate general work expression of Reference 1 can be made.

B. Buckling

For the post failure buckling region the circumferential parameter n takes on great importance and the simplification given by equation [12] cannot be employed. Instead, the plastic work has to be computed from the more general integral expressions given in Reference 1. We use equation [40] in the "Errata and Addendum of Reference 1 and assume a post failure deflection pattern of the form

$$\bar{f}(x', \phi) = \sin \pi x' e^{-k\phi} \cos n\phi \quad \text{for } 0 < \phi < \pi$$
$$= \sin \pi x' e^{-k(2\pi - \phi)} \cos n(2\pi - \phi) \quad \text{for } \pi < \phi < 2\pi$$

[23]

and an impulse distribution the same as before, i.e.

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$$I = (\overline{I}_\epsilon)\, \mathcal{E}^{-\beta \varphi} \quad (\beta = 2)$$

The shell used was made of aluminum and was
in diameter, 9" long, and .006" thick. A yield stress of 15,000 psi was
assumed. Numerical integration of equation [40] of Reference 1 was per-
formed assuming a perfectly plastic material and $\mathcal{R} = \frac{1}{4}$, $n = 5$. The
value of $n = 5$ was obtained from equation [10] and compared well with
the experimental result. The calculated peak impulse per unit area to
produce $w_0/a \approx .5$ is 21 psi milliseconds. The experiments showed
that the blast load which produced an incident impulse of 21.1 psi milli-
seconds and reflected impulse of 46.8 gave $w_0/a \approx 0.5$.

VI. Conclusions

One cannot apriori assume the type of deformation into which a shell
will deform under blast load unless some pre-failure computations are made
to determine whether the shell will collapse or buckle. Schuman's experi-
ments[2] point out that these two types of patterns can exist. The compu-
tations and experimental comparisons given in this report indicate that

1. It can be predicted whether a shell will fail in buckling or
   collapse by use of equation [8] and [9] of this report.

2. Once the failure type is known the order of magnitude of the
   impulse and energy to produce a given plastic deformation can be
   computed for side-on lateral loads by assuming a collapse pattern
   of the form of equation [13] for collapse failure and a buckled
   pattern of the form [23] for post buckling behavior in the plastic
   region.

The equations given in Reference 1 have been programmed for the BRL com-
puter. This will enable future computations with more complicated de-
flection patterns.

REFERENCES

1. J. E. Greenspon, "Elastic and Plastic Behavior of Cylindrical Shells Under Dynamic Loads Based on Energy Criteria," J G Engineering Research Associates," Contract No. DA 36-034-ORD-3081 RD, Tech. Rep. No. 3, Feb., 1963.)Under contract with Ballistic Research Laboratories, Aberdeen Proving Ground)

2. W. J. Schuman, Jr., "The Response of Cylindrical Shells to External Blast Loading," Ballistic Research Laboratories, Aberdeen Proving Ground, Memorandum Report No. 1461, March, 1963.

3. W. Flugge, "Stresses in Shells," Springer Verlag, 1962, pp. 108-110, 131.

4. V. V. Novozhilov, "The Theory of Thin Shells," Translated by P. G. Lowe, edited by J. R. M. Radok, P. Noordhoff Ltd., Grongingen, The Netherlands, 1959, p. 99, 169.

5. S. Timoshenko, "Theory of Elastic Stability," McGraw Hill Book Company, 1936, First Edition, Eighth Impression.

6. Ibid, p. 450.

7. T. E. Reynolds, "Inelastic Lobar Buckling of Cylindrical Shells Under External Hydrostatic Pressure," David Taylor Model Basin Report 1392, August, 1960.

8. B. O. Almroth, "Buckling of a Cylindrical Shell Subjected to Nonuniform External Pressure," Journal of Applied Mechanics, Dec., 1962, p. 675.
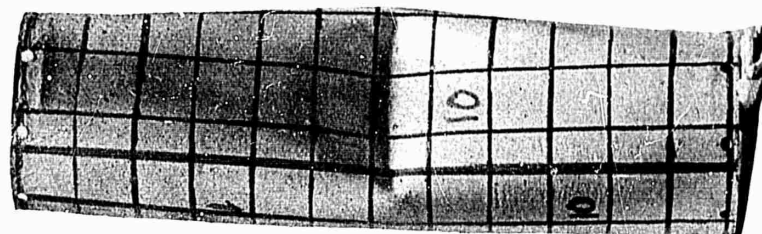
Fig. 1b. Collapse

Fig.1. Types of Failure (from Schuman's experiments, see Ref. 2)
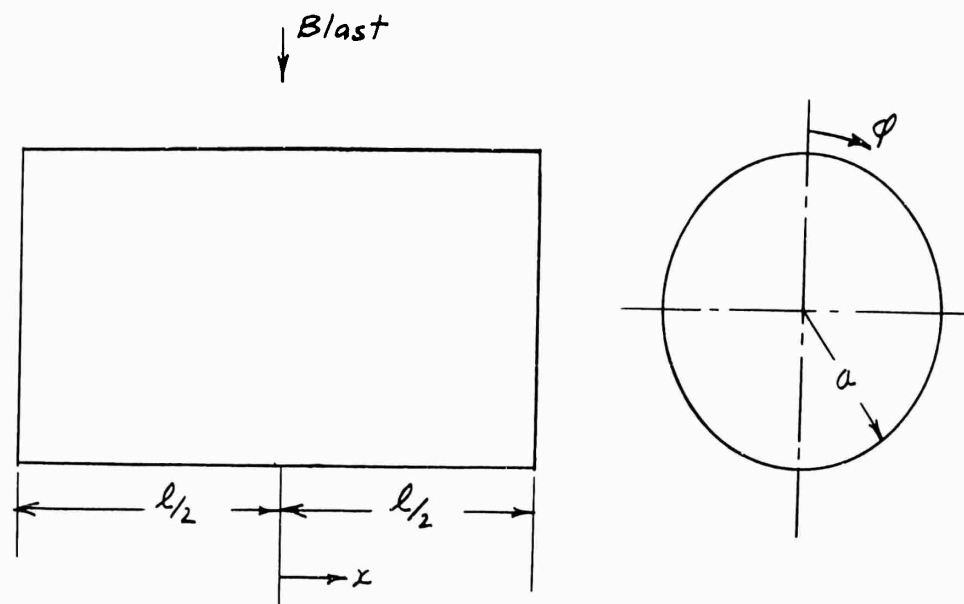
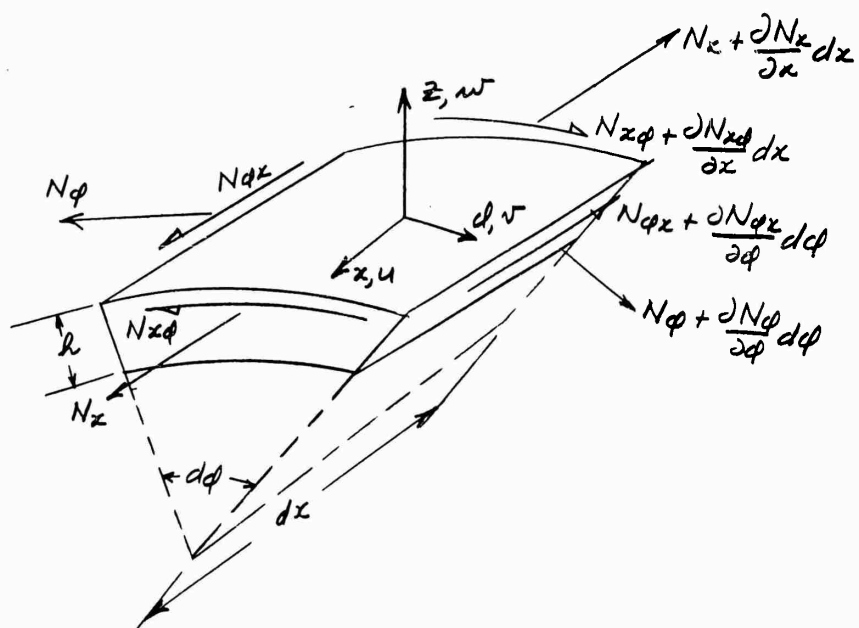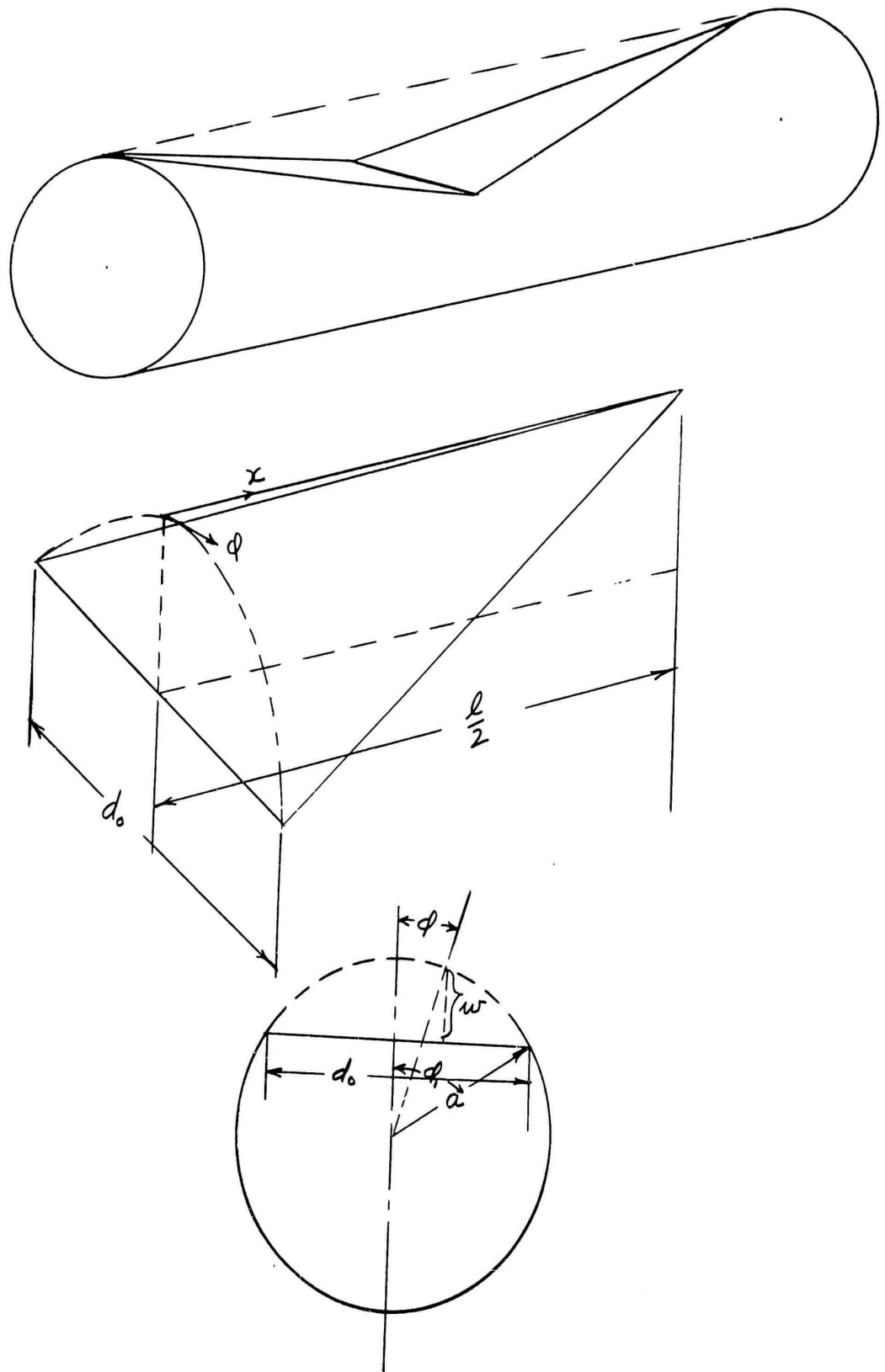Fig. 1a. Buckling

Fig. 2 Origin of Coordinates
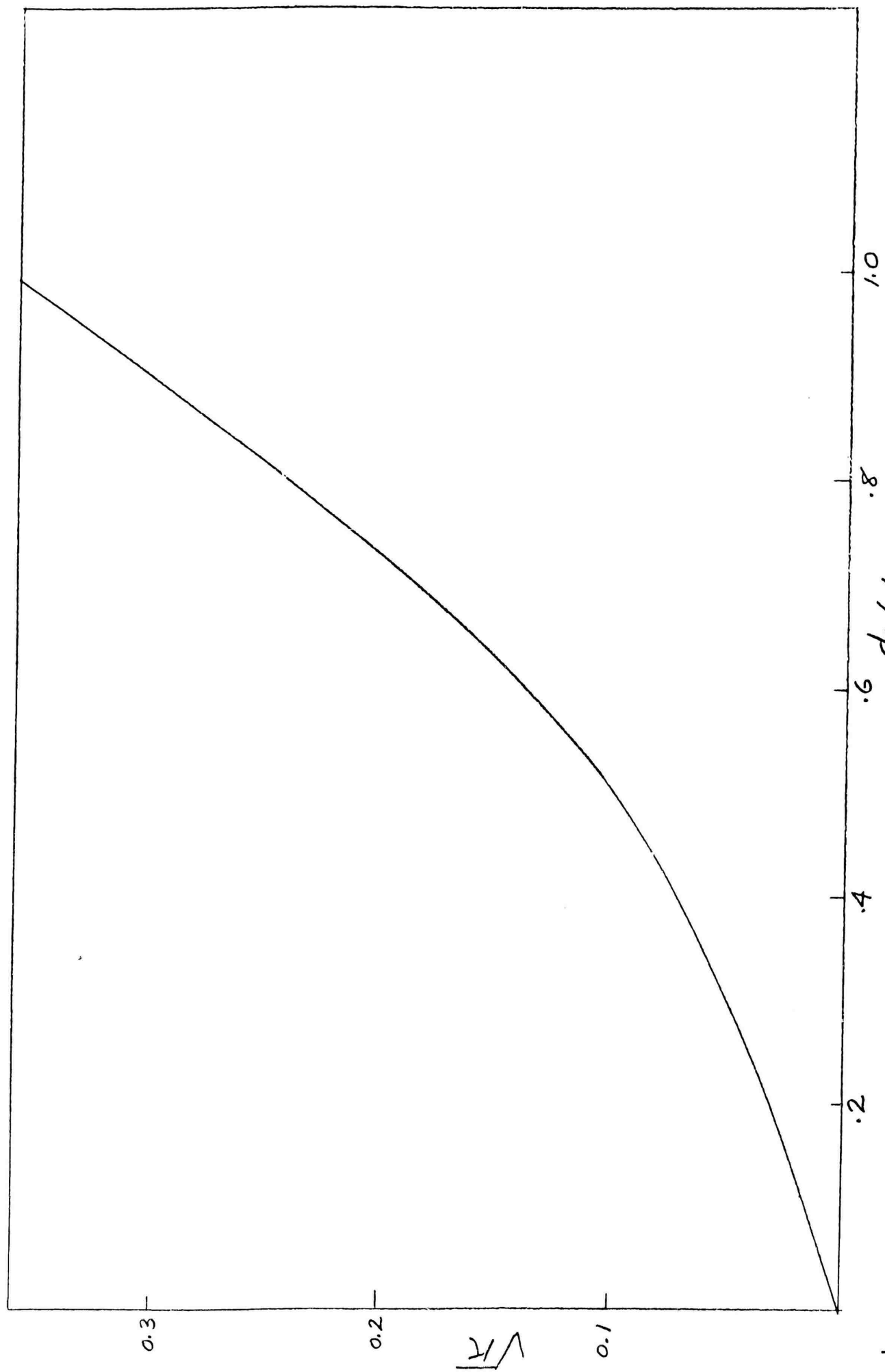

Fig. 3 Shell Element

Fig. 4 Collapse Pattern

Fig. 5 Plot of $\sqrt{\kappa}$ as a Function of d_0/d